X GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND. SOLUTIONS

1. (N.Moskvitin, V.Protasov) (8) A right-angled triangle ABC is given. Its cathetus AB is the base of a regular triangle ADB lying in the exterior of ABC, and its hypothenuse AC is the base of a regular triangle AEC lying in the interior of ABC. Lines DE and AB meet at point M. The whole configuration except points A and B was erased. Restore the point M.

   Solution. Since $\angle DAB = \angle EAC = 60°$, we have $\angle DAE = \angle BAC$, therefore triangles ADE and ABC are equal and $\angle ADE = 90°$. Thus triangleADM is right-angled with $\angle A = 60°$. Hence $AD = AB = AM/2$ (fig.1), i.e. M is the reflection of A with respect to B.

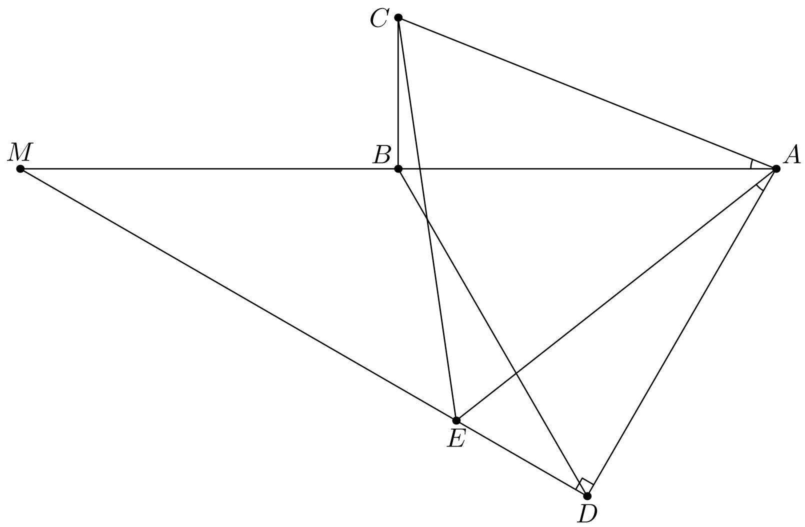

Fig.1

2. (K.Knop) (8) A paper square with sidelength 2 is given. From this square, can we cut out a 12-gon having all sidelengths equal to 1, and all angles divisible by 45°?

   Solution. Yes, see. fig.2. Points A, B, C, D lying on the medial lines of the given square are the vertices of the square with the side equal to $\sqrt{2} - 1$.

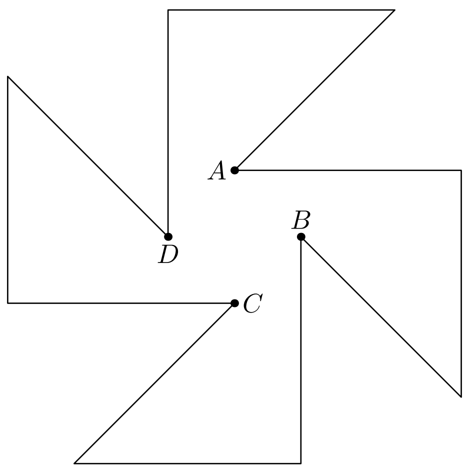

Fig.2

3. (N.Moskvitin) (8) Let ABC be an isosceles triangle with base AB. Line ℓ touches its

circumcircle at point B. Let CD be a perpendicular from C to ℓ, and AE, BF be the altitudes of ABC. Prove that D, E, F are collinear.

Solution. Let CH be the third altitude of the triangle. Since $\angle CBD = \angle CAB = \angle CBH$, the triangles CBD and CBH are equal, i.e. $BD = BH$. Also EH is the median of right-angled triangle AEB, thus $EH = HB = BD$ and $\angle BEH = \angle EBH = \angle EBD$. Therefore $EDBH$ is a parallelogram (fig.3) and $DE \parallel AB$. Since EF also is parallel to AB, lines DE and EF coincide.

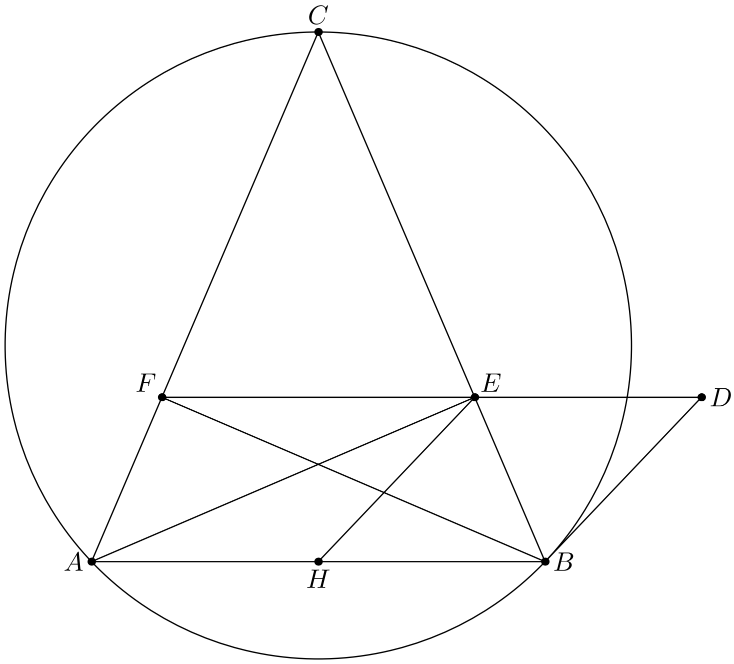

Fig.3

4. (B.Frenkin) (8) A square is inscribed into a triangle (one side of the triangle contains two vertices and each of two remaining sides contains one vertex). Prove that the incenter of the triangle lies inside the square.

   Solution. Let ABC be a triangle with incenter I, let vertices K and L of inscribed square lie on side AB, vertex M lie on AC and vertex N lie on BC (obviously angles A and B are equal). Take a perpendicular IH from I to AB and a segment DE passing through I parallel to AB with endpoints D and E lying on AC and BC respectively. We have to prove that $DE > IH$ and $H \in KL$. The first assertion is true because $IH = r$ and $DE > 2r$, where r is the radius of the incircle. Now let the extension of IH beyond I meet one of the sides of ABC at point F. We can suppose that $F \in AC$. Then H and K lie on the same side of L. Take a line passing through F, parallel to AB and intersecting BC at point G. It is sufficient to prove that $FG < FH$: then I and L lie on the same side of K and $I \in KL$.

   Note that FH contains a diameter of the incircle, thus F lies outside the incircle and $FH > 2r$. The perpendicular through F doesn't intersect the incircle. Therefore the touching points of AC and BC with the incircle lie between FG and AB. Hence the corresponding chord is greater than FG. Since it is less than $2r$, we have $FG < 2r < FH$, q.e.d.

   Comment. We see from the solution that a square can be replaced by a rectangle such that its greater side lies on the base of the triangle and is not greater than doubled smaller side.

5. (B.Frenkin) (8) In an acute-angled triangle ABC, AM is a median, AL is a bisector and AH is an altitude (H lies between L and B). It is known that $ML = LH = HB$. Find the ratios of the sidelengths of ABC.

   Answer. $AB : AC : BC = 1 : 2 : \frac{3\sqrt{2}}{2}$.

   Solution. By the property of the bisector $AC : AB = LA : LB = 2 : 1$. This follows also from the property of the median: take on the extension of AB beyond B segment $BD = AB$. Then BC is a median of the triangle ADC, and since $AL : LB = 2 : 1$, we obtain that AL also lies on a median. But AL is the bisector, therefore $AC = AD = 2AB$. Now by the Pythagor theorem we have: $AC^2 - AH^2 = AB^2 - BH^2$, or $4AB^2 - 25BH^2 = AB^2 - BH^2$, thus $AB = 2\sqrt{2}BH$ and $BC : AB = 6BH : AB = \frac{3\sqrt{2}}{2}$.

6. (A.Zaslavsky) (8–9) Given a circle with center O and a point P not lying on it. Let X be an arbitrary point of this circle, and Y be a common point of the bisector of angle POX and the perpendicular bisector to segment PX. Find the locus of points Y.

   Answer. The line perpendicular to ray OP and meeting it at the point on the distance from O equal to $(OP + OX)/2$.

   Solution. Let K, L be the projections of Y to OP and OX. By the definition of Y we have $YP = YX$ and $YK = YL$. Thus triangles YKP and YLX are equal i.e. $XL = PK$. Also $OL = OK$. Since the lengths of segments OP and OX are not equal, one of them is equal to the sum of OK and KP, and the second one is equal to their difference. Therefore $OK = (OP + OX)/2$ (fig.6). It is evident that the sought locus contains all points of the line.

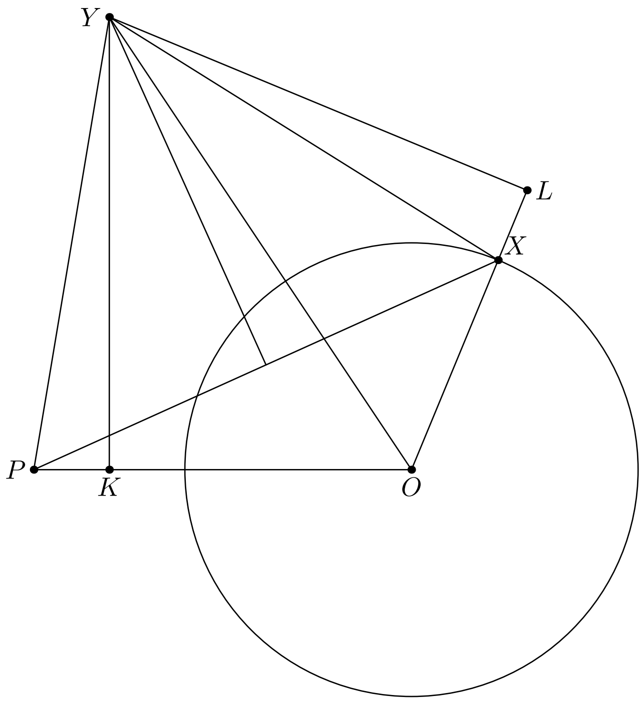

Fig.6

7. (V.Rumyantsev) (8–9) A parallelogram $ABCD$ is given. The perpendicular from C to CD meets the perpendicular from A to BD at point F, and the perpendicular from B

to AB meets the perpendicular bisector to AC at point E. Find the ratio in which side BC divides segment EF.

Answer. 1:2.

Solution. Let K be the reflection of A wrt B. Then E is the circumcenter of triangle ACK. On the other hand, since $BKCD$ is a parallelogram, we have $AF \perp CK$ and F is the orthocenter of triangle ACK. Therefore the median CB divides EF in ratio 1:2 (fig.7).

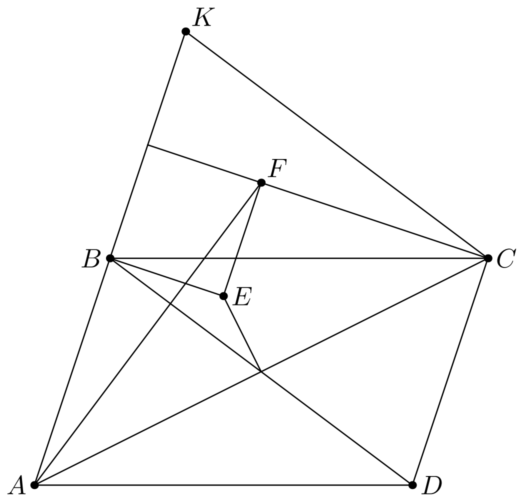

Fig.7

8. (R.Sadykov) (8–9) Given a rectangle $ABCD$. Two perpendicular lines pass through point B. One of them meets segment AD at point K, and the second one meets the extension of side CD at point L. Let F be the common point of KL and AC. Prove that $BF \perp KL$.

   First solution. Since $\angle ABK = \angle CBL$, triangles ABK and CBL are similar. Thus triangles ABC and KBL are also similar and $\angle BKF = \angle BAF$. Therefore quadrilateral $ABFK$ is cyclic and $\angle BFK = 90°$ (fig.8).

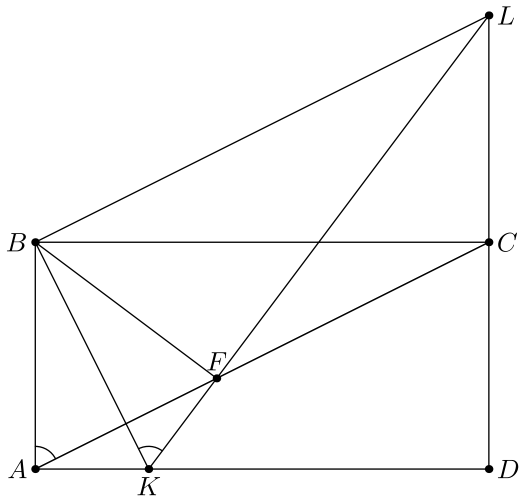

Fig.8

   Second solution. Note that point B lies on the circumcircle of triangle KLD. Points A and C are the projections of B to lines KD and DL. Thus by the Simson theorem the projection of B to KL lies on AC, i.e, coincide with F q.e.d.

9. (D.Shvetsov) (8–9) Two circles ω_1 and ω_2 touching externally at point L are inscribed into angle BAC. Circle ω_1 touches ray AB at point E, and circle ω_2 touches ray AC at point M. Line EL meets ω_2 for the second time at point Q. Prove that $MQ \parallel AL$.

   Solution. Let N be the second common point of ω_1 and AL (fig.9). Then the composition of the reflection in AL and the homothety with center A transforms arc NE to arc LM. Therefore angles ALE and MQE are equal, which yields the assertion of the problem.

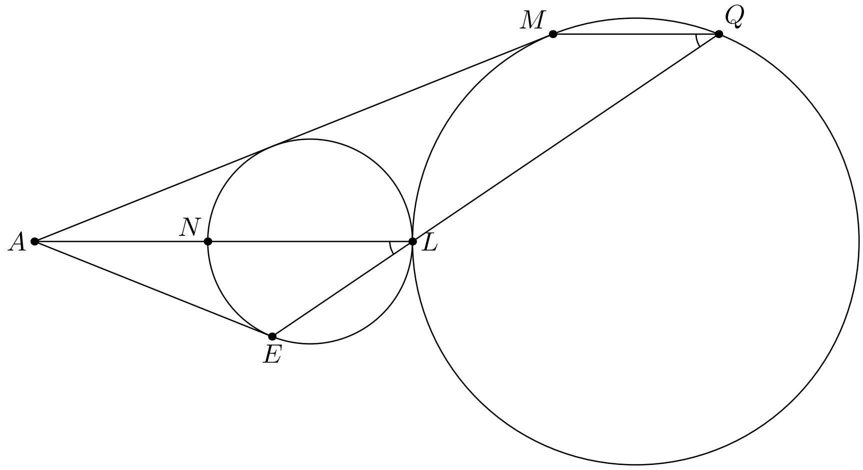

Fig.9

10. (M.Kungozhin) (8–9) Two disjoint circles ω_1 and ω_2 are inscribed into an angle. Consider all pairs of parallel lines l_1 and l_2 such that l_1 touches ω_1, and l_2 touches ω_2 (ω_1, ω_2 lie between l_1 and l_2). Prove that the medial lines of all trapezoids formed by l_1, l_2 and the sides of the angle touch some fixed circle.

   Solution. Let O_1, O_2 be the centers of the given circles, r_1, r_2 be their radii, O be the midpoint of O_1O_2, l_1' be the line parallel to l_1 and passing through O_1, l_2' be the reflection of l_1' in the medial line (fig.10). Then the distance from O_2 to l_2' is equal to $|r_2 - r_1|$. Using the homothety with center O_1 and coefficient $1/2$, we obtain that the distance d from O to the medial line is equal to $|r_2 - r_1|/2$, i.e. all medial lines touch the circle with center O and radius d.

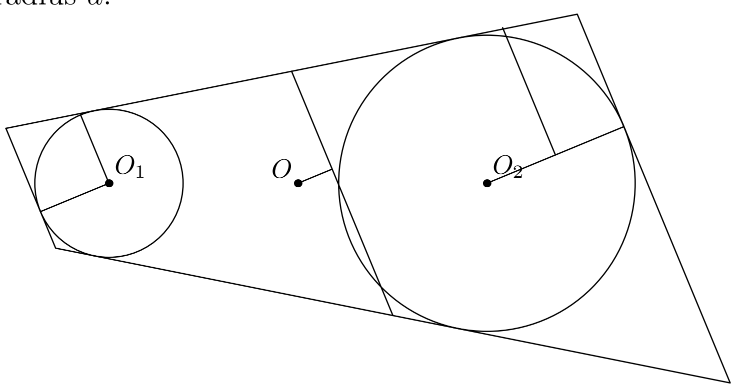

Fig.10

11. (M.Plotnikov) (8–9) Points K, L, M and N lying on the sides AB, BC, CD and DA of a square $ABCD$ are vertices of another square. Lines DK and NM meet at point E, and lines KC and LM meet at point F. Prove that $EF \parallel AB$.

   Solution. Denote the common points of lines MN and LM with AB as P and Q respectively. Triangles AKN, BLK, CML and DMN are equal by the hypothenuse and the acute angle. Let $AK = a$ and $BK = b$, then $BL = CM = DN = a$, $CL = MD = NA = b$. Since triangles PKN and QLK are right-angled, we have $PA \cdot a = b^2$ and $BK \cdot b = a^2$. The similarity of triangles PEK and DEM implies that $KE/DE = (a + b^2/a)/b = (a^2 + b^2)/ab$, but the similarity of QFK и CFM implies that $FK/CF = (b + a^2/b)/a = (a^2 + b^2)/ab$. Thus $KE/DE = FK/CF$ and $EF \parallel AB$, q.e.d.

12. (I.Makarov) (9–10) Circles ω_1 and ω_2 meet at points A and B. Let points K_1 and K_2 of ω_1 and ω_2 respectively be such that K_1A touches ω_2, and K_2A touches ω_1. The circumcircle of triangle K_1BK_2 meets lines AK_1 and AK_2 for the second time at points L_1 and L_2 respectively. Prove that L_1 and L_2 are equidistant from line AB.

Solution. Since $\angle K_1AB = \angle AK_2B$, $\angle K_2AB = \angle AK_1B$, triangles AK_1B and K_2AB are similar (fig.12). Using the sinus theorem we obtain:

$$\frac{\sin \angle K_1AB}{\sin \angle K_2AB} = \frac{AK_1}{AK_2} = \frac{AL_2}{AL_1},$$

which is equivalent to the assertion of the problem.

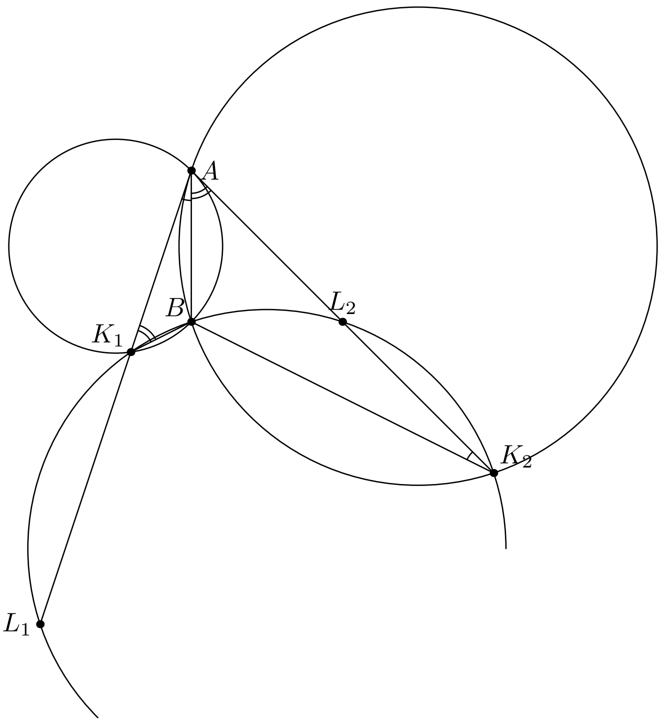

Fig.12

13. (D.Prokopenko, D.Shvetsov) (9–10) Let AC be a fixed chord of a circle ω with center O. Point B moves along the arc AC. A fixed point P lies on AC. The line passing through P and parallel to AO meets BA at point A_1; the line passing through P and parallel to CO meets BC at point C_1. Prove that the circumcenter of triangle A_1BC_1 moves along a straight line.

Solution. Let Q be the second common point of line AC and circle A_1PC_1. Then $\angle QA_1C_1 = \angle QPC_1 = \angle QCO = \angle QAO = \angle APA_1 = \angle QCA_1$. Therefore $QA_1 = QC_1$ and $\angle A_1QC_1 = \angle AOC = 2\angle A_1BC_1$, i.e. Q is the circumcenter of triangle A_1BC_1 (fig.13). Thus this circumcenter moves along line AC.

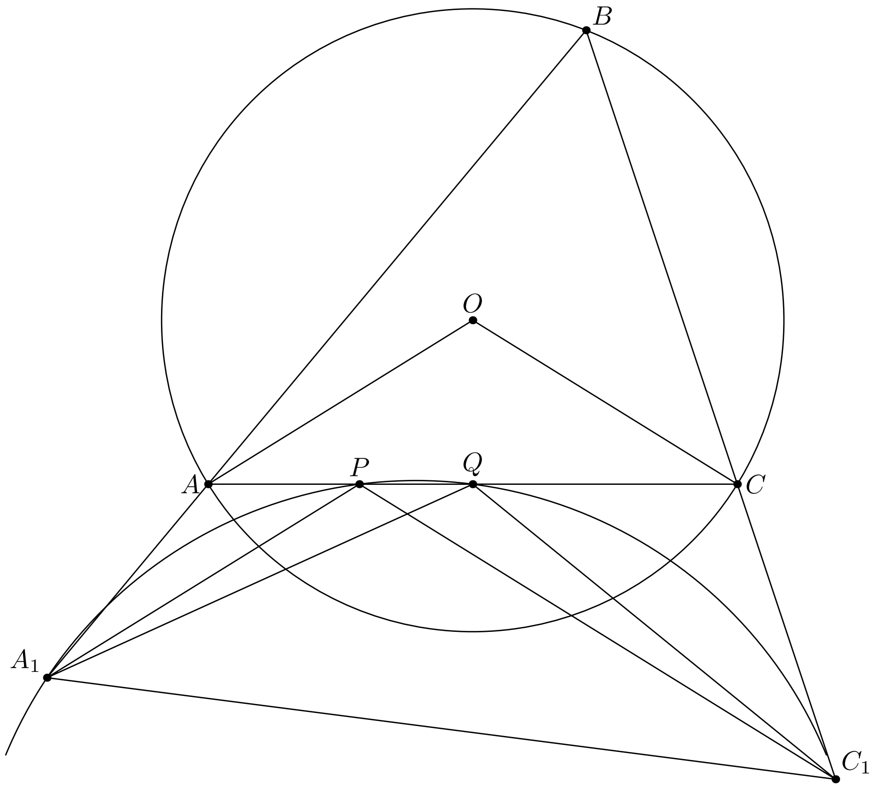

Fig.13

14. (Folklore) (9–11) In a given disc, construct a subset such that its area equals the half of the disc area and its intersection with its reflection over an arbitrary diameter has the area equal to the quarter of the disc area.

    Solution. Construct a disc, concentric to the given disc with the area equal to the half of the area of the given disc. Bisect the inner disc by an arbitrary diameter and bisect the external ring by the perpendicular diameter. Joining the half of the inner disc with the half of the ring we obtain the sought subset.

15. (9–11) Let ABC be a non-isosceles triangle. The altitude from A, the bisector from B and the median from C concur at point K.

    a) (B.Frenkin) Which of the sidelengths of the triangle is medial?

    b) (A.Zaslavsky) Which of the lengths of segments AK, BK, CK is medial?

    Answer. a) AC. b) BK.

    Solution. a) A bisector of a non-isosceles triangle lies between the corresponding altitude and median, and its altitude lies between the bisector and the smaller of two adjacent sides. Suppose that $AB < AC$. Then the bisector of angle A meets the bisector of angle B at a point, lying between K and AC. The bisector of angle C also passes through this point. Since it lies between the median and the smaller of two adjacent sides, we obtain that $AC < BC$. Thus AC is the medial side.

    Let now $AB > AC$. Then similarly the incenter lies between AK and AB, the bisector of angle C lies between CK and BC, thus $AC > BC$. Again AC is the medial side.

    b) By the condition the altitude from A lies inside the triangle, i.e. angles B and C are acute. Using the Ceva theorem we obtain that $\sin A = \cos C \mathrm{tgB}$ or $\mathrm{tgC} = \mathrm{tgB}\frac{1-\cos \mathrm{B}}{\cos \mathrm{B}}$.

Thus if $B < 60°$ then $C < B < 60° < A$, and if $B > 60°$ then $C > B > 60° > A$.

In the first case $\angle KBA < 30° < \angle KAB$ and $\angle KCB < C/2 < B/2 = \angle KBC$, therefore $KA < KB < KC$. Similarly in the second case we have $KA > KB > KC$.

Comment. Using the condition of the problem we can't define which of the sides is the greatest (the smallest), and which of segments KA, KC is the greatest (the smallest).

16. (D.Prokopenko) (9–11)Given a triangle ABC and an arbitrary point D.The lines passing through D and perpendicular to segments DA, DB, DC meet lines BC, AC, AB at points A_1, B_1, C_1 respectively. Prove that the midpoints of segments AA_1, BB_1, CC_1 are collinear.

    Solution. The circles with diameters AA_1, BB_1, CC_1 pass through the bases of the correspondent altitudes, thus the degrees of orthocenter H wrt these three circles are equal. Therefore line DH is their common radical axis and their centers are collinear.

    Comment. Applying the Menelaos theorem to triangle ABC and its medial triangle we can obtain that A_1, B_1, C_1 are also collinear.

17. (N.Moskvitin) (10–11) Let AC be the hypothenuse of a right-angled triangle ABC. The bisector BD is given, and the midpoints E and F of the arcs BD of the circumcircles of triangles ADB and CDB respectively are marked (the circles are erased). Construct the centers of these circles using only a ruler.

    Solution. We will use following well-known facts.

    1.) If two parallel lines are given then we can bisect a segment lying on one of them, using only a ruler.

    2.) If two parallel lines are given then we can construct a line parallel to them and passing through a fixed point not lying on these lines, using only a ruler.

    Note now that EF is the perpendicular bisector to BD. Thus its common points K, L with AB and BC are the vertices of a square $BKDL$. Using parallel lines BC and KD bisect segment BC. Using parallel lines AB and DL construct the line parallel to them trough the midpoint of BC. This line is the perpendicular bisector of BC, therefore it meets EF at the circumcenter of triangle BCD. The circumcenter of triangle ABD can be constructed similarly.

18. (A.Zaslavsky) (10–11) Let I be the incenter of a circumscribed quadrilateral ABCD. The tangents to circle AIC at points A, C meet at point X. The tangents to circle BID at points B, D meet at point Y. Prove that X, I, Y are collinear.

    Solution. Let J be the second common point of circles AIC and BID. The inversion wrt the incircle of $ABCD$ transforms A, B, C, D to the vertices of a parallelogram, also it transforms J to the center of this parallelogram. Therefore $AJ/CJ = AI/CI$, i.e line IJ is the symedian of triangle AIC, thus this line passes through X. Similarly it passes through Y.

19. (V.Yassinsky) (10–11) Two circles ω_1 and ω_2 touch externally at point P. Let A be a point of ω_2 not lying on the line through the centers of the circles, and AB, AC be the tangents to ω_1. Lines BP, CP meet ω_2 for the second time at points E and F. Prove that line EF, the tangent to ω_2 at point A and the common tangent at P concur.

Solution. The homothety with center P transforms B, C to E, F. Thus it transforms A to the pole of line EF wrt ω_2, i.e. the pole of EF lies on AP, which is equivalent to the assertion of the problem.

20. (N.Beluhov) (10–11) A quadrilateral $KLMN$ is given. A circle with center O meets its side KL at points A and A_1, side LM at points B and B_1, etc. Prove that if the circumcircles of triangles KDA, LAB, MBC and NCD concur at point P, then

    a) the circumcircles of triangles KD_1A_1, LA_1B_1, MB_1C_1 and NC_1D_1 also concur at some point Q;

    b) point O lies on the perpendicular bisector to PQ.

    Solution. Let $A_1'B_1'$ be a variable chord in the circle, equal to A_1B_1, i.e. obtained from A_1B_1 by rotation with center O. Easy computation of angles shows that the circle (LAB) is in fact the locus of the intersection $K' = AA_1' \cap BB_1'$ as $A_1'B_1'$ moves around the circle. Thus, since P is the intersection of four such loci, the lines AP, BP, CP and DP must intersect the circle in four points A', B', C', D', forming a quadrilateral equal to $A_1B_1C_1D_1$. Consider the rotation with center O, sending $A'B'C'D'$ to $A_1B_1C_1D_1$, and let it send P to some point Q. Then the lines A_1Q, B_1Q, C_1Q and D_1Q will intersect the circle in four points, forming a quadrilateral, equal to $ABCD$. The same loci argument, applied to the circumcircles of $\triangle KD_1A_1, \triangle LA_1B_1, \triangle MB_1C_1$ and $\triangle NC_1D_1$, shows that they are concurrent in Q. Also, since OQ is the image of OP under the rotation, we have $OP = OQ$, and (b) also follows.

21. (N.Poljansky, D.Skrobot) (10–11) Let $ABCD$ be a circumscribed quadrilateral. Its incircle ω touches sides BC and DA at points E and F respectively. It is known that lines AB, FE and CD concur. The circumcircles of triangles AED and BFC meet ω for the second time at points E_1 and F_1. Prove that $EF \parallel E_1F_1$.

    Solution. Let R be the common point of BC and AD. Then R and the touching points P и Q of the incircle with two remaining sides are collinear.

    Let EE_1 meet AD at point M. Consider three circles: the incircle of $ABCD$, AED and AID, where I is the incenter of $ABCD$. It is clear that the radical axis of AID and the incircle is the medial line of triangle FPQ. Since two remaining radical axes meet at M we obtain that $RM = MF$ (fig.21).

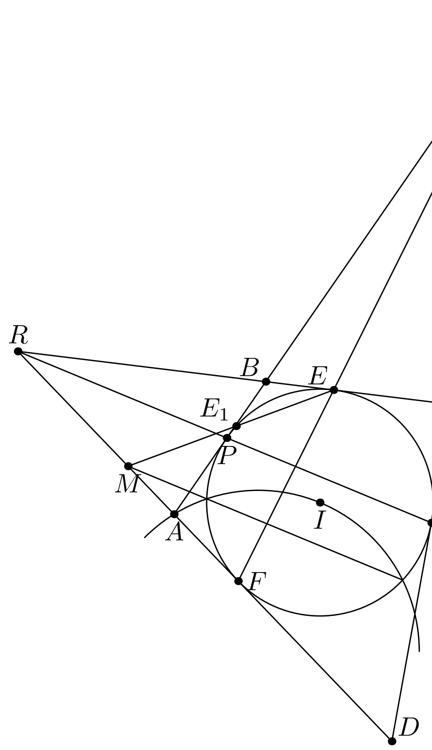

Fig.21

Similarly, FF_1 meets BC at point N, such that $RN = NE$. Therefore lines EE_1 and FF_1 are symmetric wrt the bisector of angle ERF. Thus points E_1 and F_1 are also symmetric and EFF_1E_1 is an isosceles trapezoid.

22. (A.Blinkov) (10–11) Does there exist a convex polyhedron such that it has diagonals and each of them is shorter than each of its edges?

    Solution. Yes, take a regular triangle ABC with side equal to 1 and two points S_1, S_2, symmetric wrt its plane and such that $S_1S_2 < S_1A = S_1B = S_1C < 1$. It is evident that the unique diagonal S_1S_2 of the obtained polyhedron is shorter than each of its edges.

23. (A.Akopyan) (11) Let A, B, C and D be a triharmonic quadruple of points, i.e

$$AB \cdot CD = AC \cdot BD = AD \cdot BC.$$

Let A_1 be a point distinct from A such that the quadruple A_1, B, C and D is triharmonic. Points B_1, C_1 and D_1 are defined similarly. Prove that

a) A, B, C_1, D_1 are concyclic;

b) the quadruple A_1, B_1, C_1, D_1 is triharmonic.

Solution. a) Consider three spheres touching the given plane at points A, B, C and externally touching each other. If the radii of these spheres are equal to x, y, z, then $AB = 2\sqrt{xy}$ etc. Thus there exist two spheres touching the plane at points D and D_1 and touching three given spheres. Therefore we can construct eight spheres a, b, c, d, a_1, b_1, c_1, d_1, touching the plane at A, B, C, D, A_1, B_1, C_1, D_1, and such that a and a_1 touch b, c, d etc.

Take an inversion of the space with the center at the touching point of c and d. It transforms these two spheres to two parallel planes, and the given plane, a and b will be transformed to three equal mutually touching spheres lying between these two planes. The images of c_1 and d_1 have to touch these three spheres, also each of these two spheres touches one of the planes, therefore they are symmetric wrt the plane containing the centers of three remaining spheres. Thus the images of A, B, C_1, D_1 are complanar and these points are concyclic.

b) Consider now an inversion with center D. It transforms d to the plane parallel to ABC, and the images of a, b, c are three equal mutually touching spheres. Therefore their touching points with the plane are the vertices of a regular triangle, and the image of D_1 is the center of this triangle. The images of A_1, B_1, C_1 are the vertices of a regular triangle with the same center, i.e. quadruple A_1, B_1, C_1, D_1 is triharmonic.

24. (F.Nilov) (11) A circumscribed pyramid $ABCDS$ is given. The opposite sidelines of its base meet at points P and Q in such a way that A and B lie on segments PD and PC respectively. The inscribed sphere touches faces ABS and BCS at points K and L. Prove that if PK and QL are complanar then the touching point of the sphere with the base lies on BD.

First solution. Since P, Q, K and L are complanar, segments PL and QP meet at point R lying on BS. Let T be the touching point of the insphere with the base of the pyramid. Note that triangles QBK and QB are equal and triangles PBL and PBT are equal (by the equality of the correspondent tangents). Similarly triangles RKB and RLB are equal. Thus $\angle QTB = \angle QKB = \angle PLB = \angle PTB$. But in the circumscribed pyramid $\angle CTQ = \angle PTA$ and $\angle CTD + \angle ATB = 180^{circ}$, therefore $\angle PTB = 180°$.

Second solution. Consider a projective map saving the insphere and transforming PQS to the infinite plane. It transforms the pyramid to the infinite prism, and by the complanarity of PK and QL we obtain that the facets of this prism passing through AB and BC form equal angles with plane $ABCD$. Thus the prism is symmetric wrt the plane passing through BD and perpendicular to $ABCD$. It is clear that the touching point of the insphere with the base lies on the plane of symmetry.