XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 8 grade

Ratmino, 2016, July 31

1. (Yu.Blinkov) An altitude AH of triangle ABC bisects a median BM. Prove that the medians of triangle ABM are sidelengths of a right-angled triangle.

   Solution. Let AH and BM meet at point K, let L be the midpoint of AM, and let N and P be the projections of L and M respectively to BC (fig.8.1). Since K is the midpoint of BM, it follows that KH is a midline of triangle BMP, i.e. $PH = HB$. On the other hand, by the Thales theorem $CP = PH$ and $PN = NH$, hence N is the midpoint of BC. Therefore NK is a medial line of triangle BMC, i.e. $NK \parallel AC$ and $ALNK$ is a parallelogram. Hence $LN = AK$. Also the median from M in triangle AMB is a midline of ABC, hence it is congruent to BN. Therefore the sides of right-angled triangle BNL are congruent to the medians of ABM.

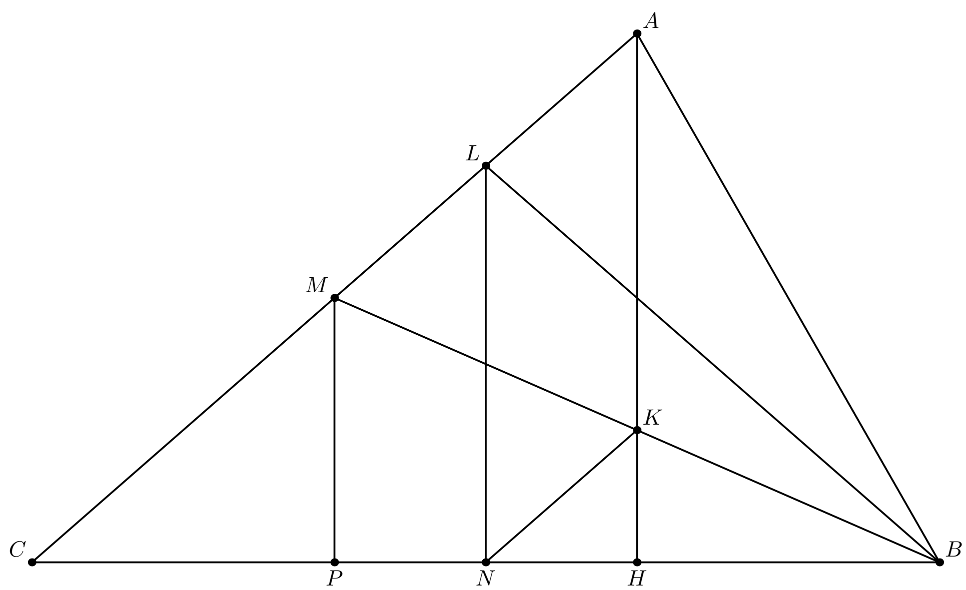

Fig. 8.1

2. (E.Bakaev) A circumcircle of triangle ABC meets the sides AD and CD of a parallelogram $ABCD$ at points K and L respectively. Let M be the midpoint of arc KL not containing B. Prove that $DM \perp AC$.

   First solution. By the assumption we obtain that $ALCB$ is an isosceles trapezoid, i.e $AL = AD$ (fig.8.2). Now AM is the bisector of isosceles triangle ALD, thus AM is also its altitude. Hence $AM \perp CD$. Similarly $CM \perp AD$. Therefore M is the orthocenter of triangle ACD and $DM \perp AC$.

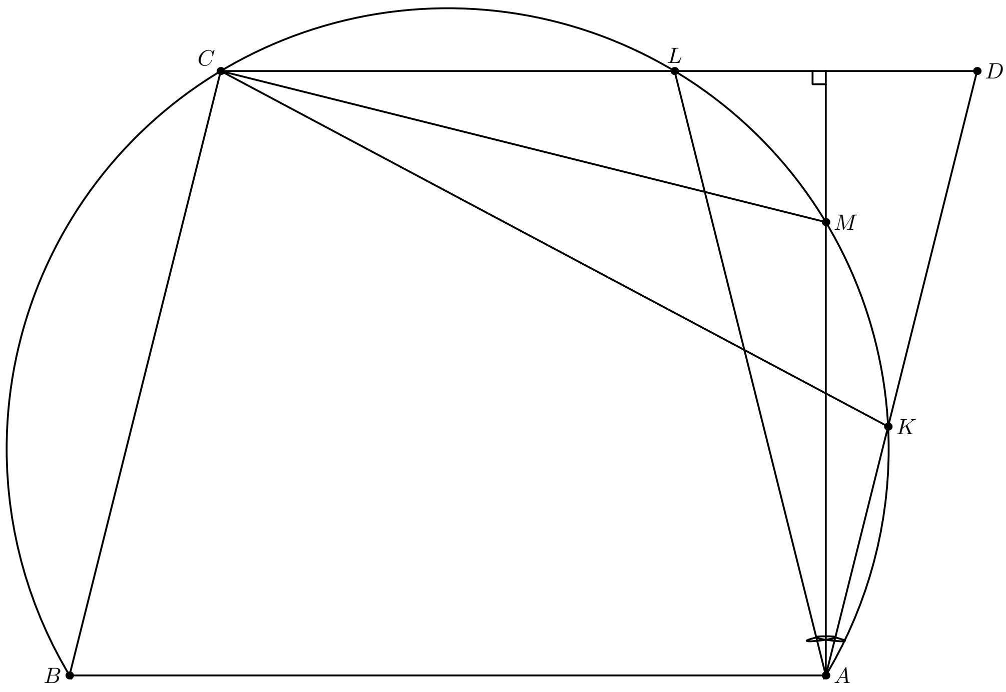

Fig. 8.2

Second solution. Consider the circumcircle of triangle ABC. The equality of angles BAK and BCL yields the equality of arcs BAK and BCL. The arcs LM and KM are also equal, and since the sum of these four arcs is the whole circle, we obtain that BM is a diameter. Then the triangles BAM and BCM are right-angled, i.e $BA^2 + AM^2 = BM^2 = BC^2 + CM^2$. Rewrite this equality as $BA^2 - BC^2 = CM^2 - AM^2$ and modify left part using the equality of opposite sides of parallelogram: $CD^2 - AD^2 = CM^2 - AM^2$. By the Carnot principle we obtain that $DM \perp AC$.

3. (D.Prokopenko) A trapezoid $ABCD$ and a line l perpendicular to its bases AD and BC are given. A point X moves along l. The perpendiculars from A to BX and from D to CX meet at point Y. Find the locus of Y.

   Answer. The line l' that is perpendicular to the bases of the trapezoid and divides AD in the same ratio as l divides CB.

   First solution. Let XU, YV be the altitudes of triangles BXC, AYD (fig.8.3.1). Then $\angle YAV = \angle BXU$ and $\angle YAD = \angle CXU$ because the sides of these angles are perpendicular. Therefore the triangle AVY is similar to XUB, and the triangle DVY is similar to XUC. From this we obtain that the ratio $AV : VD = CU : UB$ does not depend on X, i.e. Y lies on l'. It is clear that all points of this line are in the required locus.

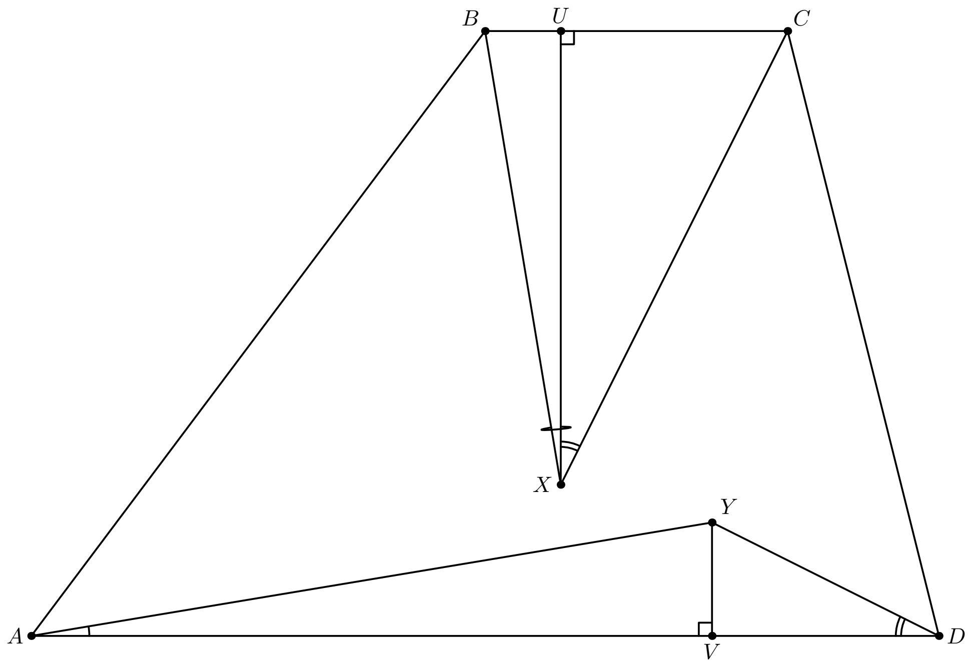

Fig. 8.3

Second solution. The locus of points with the constant difference of squares of the distances from the endpoints of a segment is a line perpendicular to this segment. Hence it is sufficient to prove that the difference $YB^2 - YC^2$ is constant.

Since the lines BX and AY are perpendicular, we have $YB^2 - AB^2 = YX^2 - AX^2$. Similarly $DC^2 - YC^2 = DX^2 - YX^2$. Summing these equalities we obtain that $YB^2 - YC^2 = (DX^2 - AX^2) + (AB^2 - DC^2)$. The first difference is constant by the definition of X. Therefore all points Y lie on the line perpendicular to BC.

Third solution. Let the lines AB and CD meet at point P. Consider the homothety with center P mapping the segment BC to AD. Let X' be the image of X. The homothety maps BX and CX to parallel lines AX' and DX'. Therefore the angles $X'AY$ and $X'DY$ are right and the quadrilateral $X'AYD$ is cyclic. We obtain also that X' moves along a fixed line l' parallel to l.

Let Q, R be the projections of X' and Y to AD (fig. 8.3.2). Since the midpoint of diameter $X'Y$ is projected to the midpoint of chord AD, we obtain by the Thales theorem that $AQ = DR$. The point Q is fixed, hence Y moves along the line passing through R and parallel to the bases.

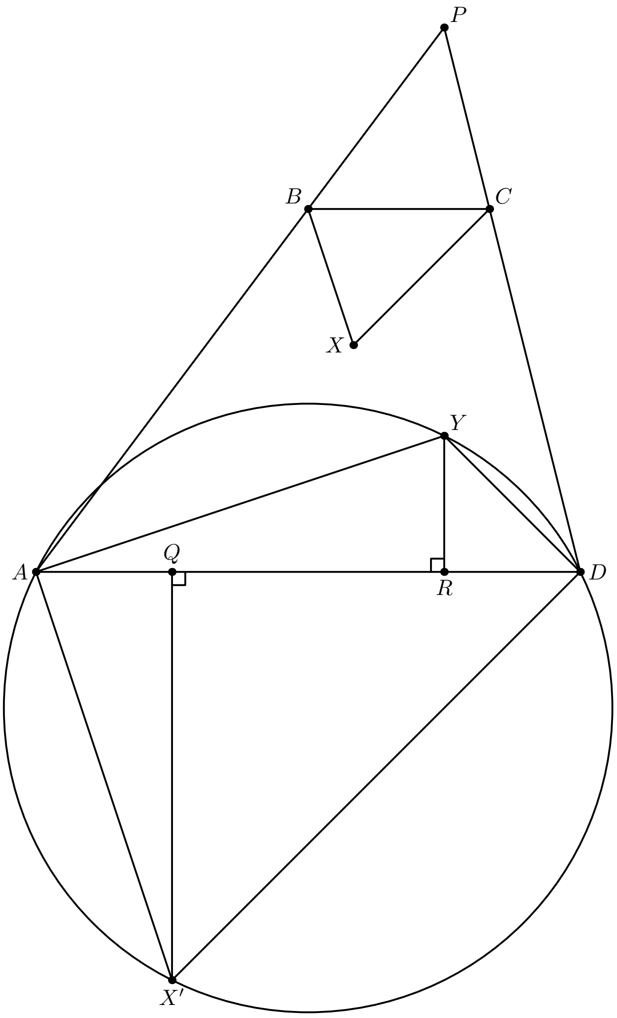

Fig. 8.3.2

4. (N.Beluhov) Is it possible to dissect a regular decagon along some of its diagonals so that the resulting parts can form two regular polygons?

   Answer. Yes, see fig.8.4

   Remark. This construction works for all regular $2n$-gons with $n \geq 3$.

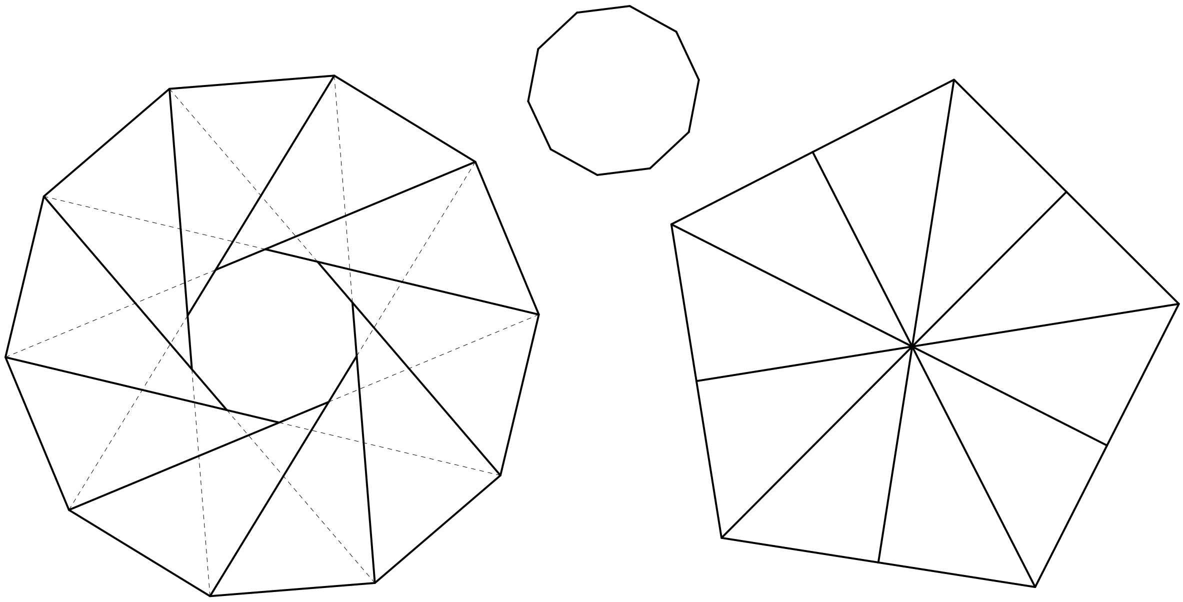

Fig.8.4

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. Second day. 8 grade

Ratmino, 2016, August 1

5. (A.Khachaturyan) Three points are marked on the transparent sheet of paper. Prove that the sheet can be folded along some line in such a way that these points form an equilateral triangle.

   Solution. Let A, B, C be the given points, AB be the smallest side of triangle ABC, D be the vertex of an equilateral triangle ABD, l be the perpendicular bisector to segment CD. Since $AD = AB \leq AC$ and $BD = AB \leq BC$, the points A, B lie on the same side from l as D. Thus if we fold the sheet along l then A and B do not move, and C maps to D.

6. (E.Bakaev) A triangle ABC with $\angle A = 60°$ is given. Points M and N on AB and AC respectively are such that the circumcenter of ABC bisects segment MN. Find the ratio $AN : MB$.

   Answer. 2.

   First solution. Let P, Q be the projections of N and the circumcenter O respectively to AB (fig.8.6). From the condition we have $MQ = QP$. On the other hand Q is the midpoint of AB, thus $BM = AP$. But in the right-angled triangle APN we have $\angle A = 60°$. Therefore $BM = AP = AN/2$.

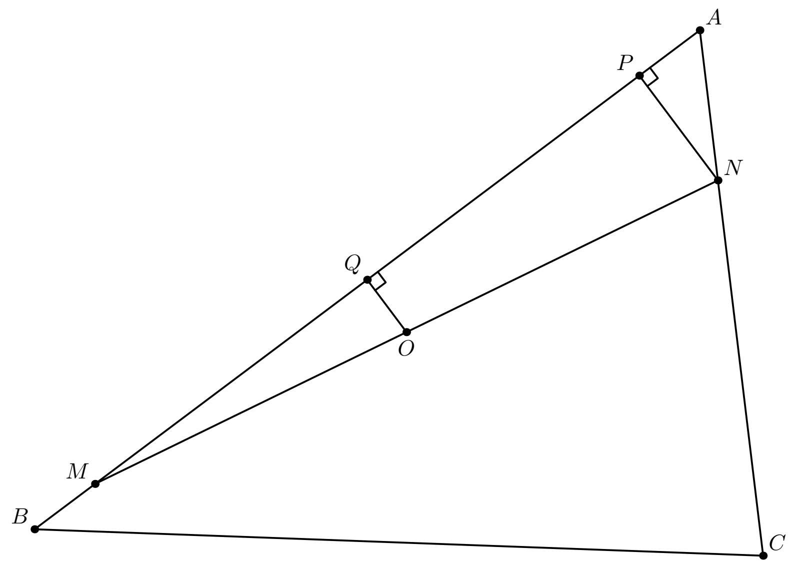

Fig. 8.6

   Second solution. Let P be the point on the circumcircle of ABC opposite to A. Since O bisects the segments AP and MN, we have that $AMPN$ is a parallelogram. The angles BAC and BMP are equal because $AC \parallel MP$. The angle ABP is right because AP is a diameter. Thus BMP is a right-angled triangle with $\angle M = 60°$, therefore $MP : MB = 2$. The segments MP and AN are the opposite sides of parallelogram, hence $AN : MB = 2$.

7. (A.Zaslavsky) Diagonals of a quadrilateral $ABCD$ are equal and meet at point O. The perpendicular bisectors to segments AB and CD meet at point P, and the perpendicular bisectors to BC and AD meet at point Q. Find angle POQ.

   Answer. $90°$.

   Solution. Since $PA = PB$ and $PC = PD$, the triangles PAC and PBD are congruent (fig.8.7). Therefore the distances from P to the lines AC and BD are equal, i.e. P lies on the bisector of some angle formed by these lines. Similarly Q also lies on the bisector of some of these angles. Let us prove that these points lie on different bisectors. The bisector of angle AOB meets the perpendicular bisector to AB at the midpoint of arc AB of the circle AOB. Also this bisector meets the perpendicular bisector to CD at the midpoint of arc CD of circle COD. These two points lie on the different sides from O, hence P lies on the bisector of angle AOD. Similarly Q lies on the bisector of angle AOB. It is evident that these bisectors are perpendicular.

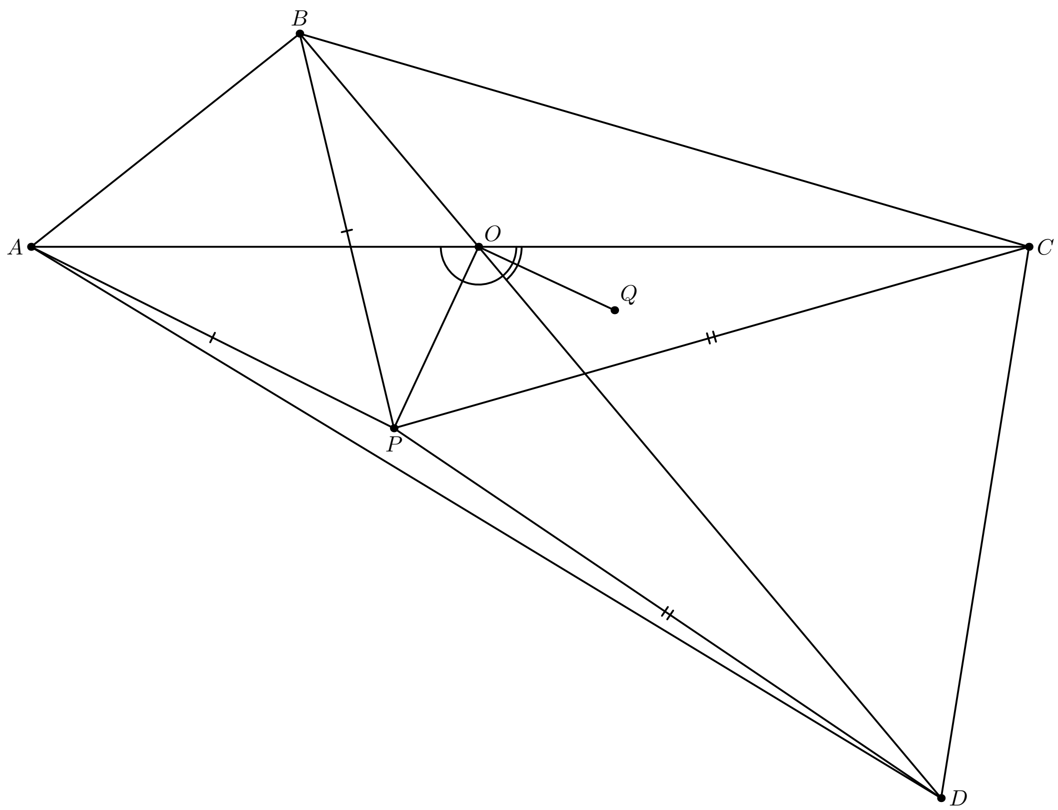

Fig. 8.7

8. (V.Protasov) A criminal is at point X, and three policemen at points A, B and C block him up, i.e. the point X lies inside the triangle ABC. Each evening one of the policemen is replaced in the following way: a new policeman takes the position equidistant from three former policemen, after this one of the former policemen goes away so that three remaining policemen block up the criminal too. May the policemen after some time occupy again the points A, B and C (it is known that at any moment X does not lie on a side of the triangle)?

Answer. No.

First solution. It is evident that all triangles formed by the policemen after the first evening are isosceles. Thus we can suppose that in the original triangle $AC = BC$. Let O, R be the circumcenter and the circumradius of triangle ABC. Then since $OC \perp AB$ and X lies inside ABC, we obtain that the projection of X to the altitude CD lies between C and D. Hence $XC^2 - XO^2 < CD^2 - DO^2 = AC^2 - AO^2$ or $XC^2 - AC^2 < XO^2 - R^2$. Similarly $O'X^2 - R'^2 < OX^2 - R^2$, where O', R' are the circumcenter and the circumradius of the new triangle formed by the policemen. Therefore the degree of X wrt the circumcircle of policemen's triangle decreases each evening and the policemen cannot occupy the initial points.

Second solution. Let A be the vertex of the triangle nearest to X, and O be the circumcenter. It is clear that X cannot lie inside the triangle OBC, i.e. A is a vertex of the new triangle containing X. Therefore the distance from X to the nearest vertex does not increase. This is also correct for the further steps. If the sequence of triangles is periodic then this distance is constant and A is the vertex of all triangles containing X. These triangles are isosceles and A is the vertex at the base, i.e. the angle at this vertex is acute. Hence one of rays BO, CO passes through the triangle. Let the extension of segment AX meet BC at point Y . Since one of rays BO, CO intersects the segment AY, we obtain that the distance XY decreases at each step, therefore the policemen cannot occupy the initial points again.

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 9 grade

Ratmino, 2016, July 31

1. (D.Shvetsov) The diagonals of a parallelogram $ABCD$ meet at point O. The tangent to the circumcircle of triangle BOC at O meets the ray CB at point F. The circumcircle of triangle FOD meets BC for the second time at point G. Prove that $AG = AB$.

   Solution. From the tangency we have $\angle FOB = \angle BCO = \angle GCA$, and since $FGOD$ is cyclic, $\angle FOB = \angle DGC$.

   We obtain that $\angle GCA = \angle DGC$, hence $AGCD$ is an isosceles trapezoid and $AG = DC = AB$.

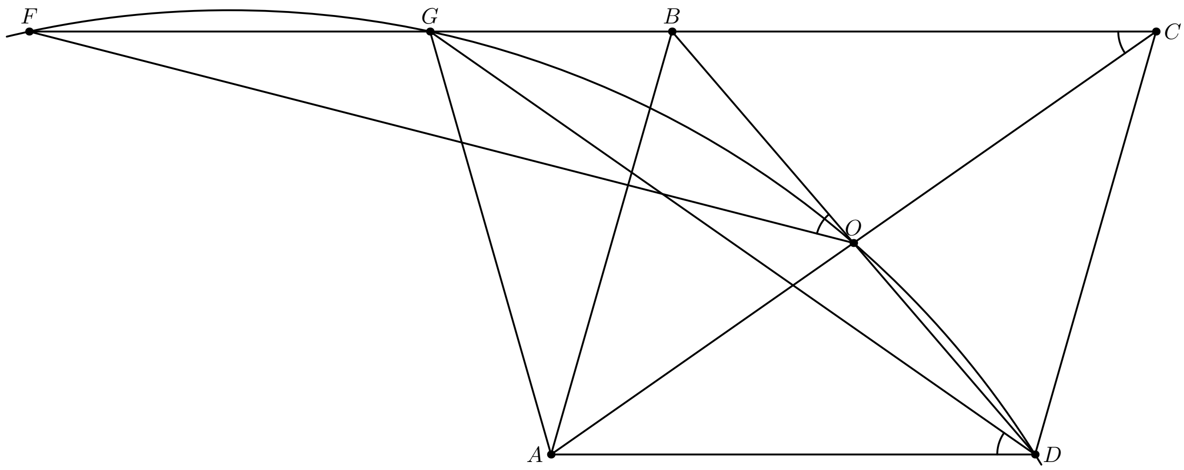

Fig. 9.1

2. (D.Khilko) Let H be the orthocenter of an acute-angled triangle ABC. Point X_A lying on the tangent at H to the circumcircle of triangle BHC is such that $AH = AX_A$ and $H \neq X_A$. Points X_B and X_C are defined similarly. Prove that the triangle $X_AX_BX_C$ and the orthotriangle of ABC are similar.

   Solution. Let O be the circumcenter of ABC (fig. 9.2). Let us prove that $AO \perp HX_A$. In fact, the translation by vector AH maps the circle ABC to the circle BHC. Hence the tangent at H is parallel to the tangent at A and perpendicular to the radius OA. Since HAX_A is an isosceles triangle, its altitude coincides with the median. Thus AO is the perpendicular bisector to HX_A. Similarly BO, CO are the perpendicular bisectors to HX_B, HX_C respectively. Therefore H, X_A, X_B, X_C lie on a circle centered at O. Now we have $\angle X_AX_CX_B = \angle X_AHX_B = \angle CHX_A + \angle X_BHC = 2(90^\circ -$

$\angle C) = \angle H_1H_3H_2$. Similarly $\angle X_AX_BX_C = \angle H_1H_2H_3$ and $\angle X_BX_AX_C = \angle H_2H_1H_3$. Since the correspondent angles of triangles $X_AX_BX_C$ and $H_1H_2H_3$ are equal, these triangles are similar.

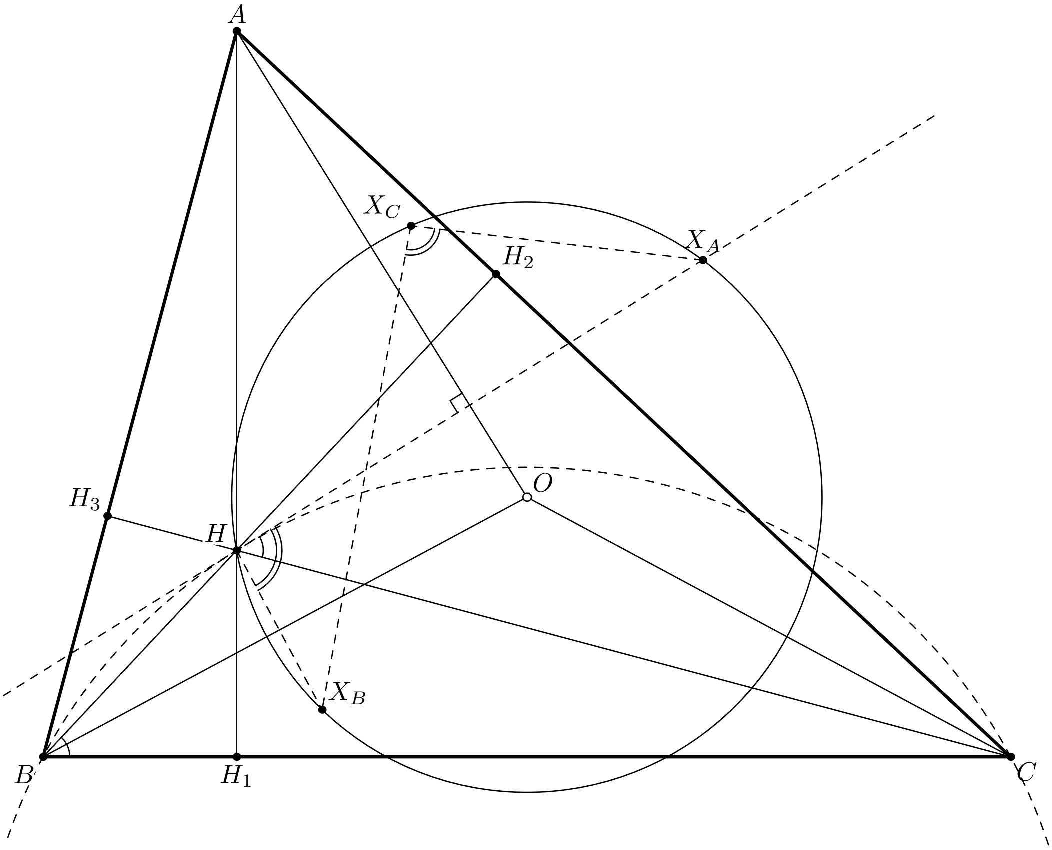

Fig. 9.2.

Remark. This solution can be modified. The midpoints of segments HX_A, HX_B, HX_C lie on the circle with diameter OH and form a triangle similar to the orthotriangle (this can be proved as above). This reasoning allows to prove a general assertion: if P and Q are isogonally conjugated, and A_1, B_1, C_1 are the projections of P to AQ, BQ and CQ, then the triangle $A_1B_1C_1$ is similar to the pedal triangle of P.

3. (V.Kalashnikov) Let O and I be the circumcenter and the incenter of triangle ABC. The perpendicular from I to OI meets AB and the external bisector of angle C at points X and Y respectively. In what ratio does I divide the segment XY?

   Answer. 1 : 2.

   First solution. Let I_a, I_b, I_c be the excenters of ABC. Then ABC and its circumcircle are the orthotriangle and the nine-points circle of triangle

$I_aI_bI_c$. Hence the circumcenter of $I_aI_bI_c$ is symmetric to I wrt O, and its circumradius is equal to the double circumradius of ABC. The triangle $A'B'C'$ homothetic to ABC with center I and coefficient 2 has the same circumcircle. The line l passing through I and perpendicular to OI carves the chord of this circle with midpoint I, the chords I_aA' and I_bB' also pass through it (fig.9.3). By the butterfly theorem I_aI_b and $A'B'$ meet l at two points symmetric about I, therefore $IX : IY = 1 : 2$.

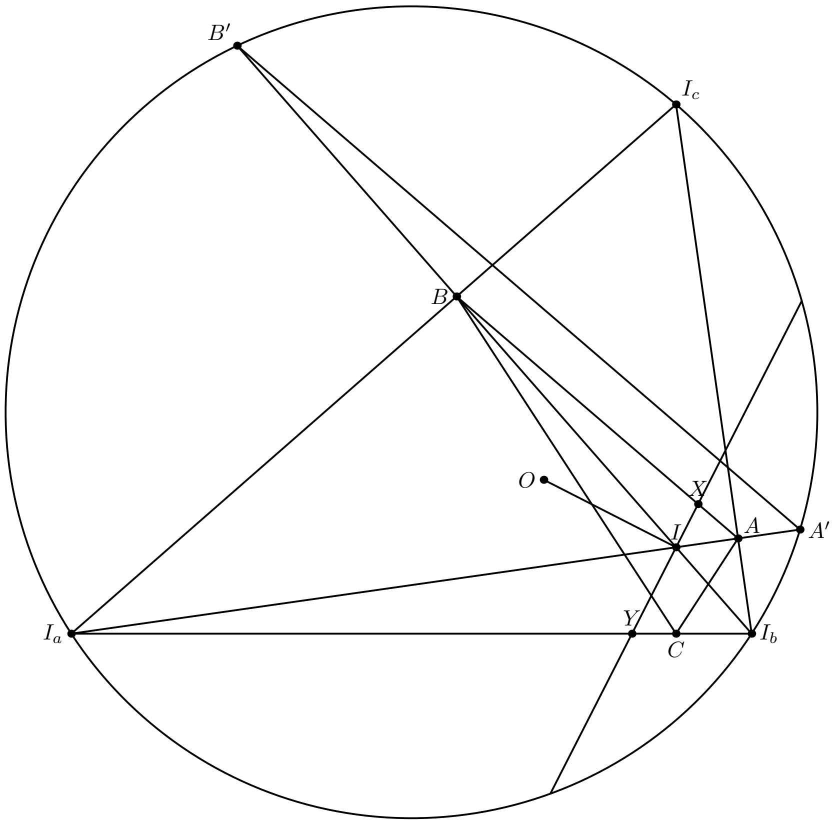

Fig. 9.3

Second solution. Consider the points such that the sum of oriented distances from them to the sidelines of ABC is equal to $3r$, where r is the radius of the incircle. Since the distance is a linear function, the locus of such points is a line passing through I. Since the sum of the projections of vector OI to the lines AB, BC, CA is zero, this line is perpendicular to OI. Since Y lies on the external bisector of angle C, the sum of distances from Y to AC and BC is zero. Thus the distance from Y to AB is equal to $3r$, i.e. $YX = 3IX$.

4. (N.Beluhov) One hundred and one beetles are crawling in the plane. Some of the beetles are friends. Every one hundred beetles can position themselves so that two of them are friends if and only if they are at the unit distance

from each other. Is it always true that all one hundred and one beetles can do the same?

Answer. No.

First solution. Let two beetles be friends if and only if they are connected by a solid line in the fig. 9.4.

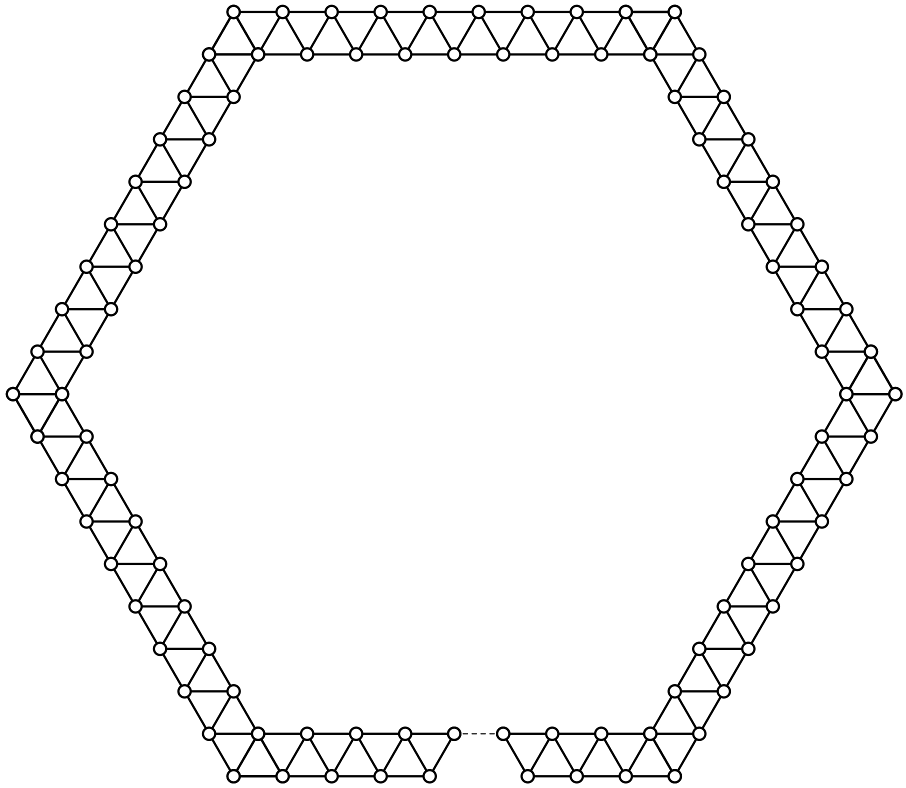

Fig. 9.4.

Suppose that all one hundred and one beetles have positioned themselves so that the only if part is satisfied (if two beetles are friends then they are the unit distance apart). If two beetles occupy the same position then the if part (if two beetles are the unit distance apart then they are friends) fails. Otherwise, friendships determine a unique structure which forces the two beetles connected by a dashed line to be the unit distance apart without being friends and the if part fails again.

If we temporarily forget about any one beetle, the structure becomes flexible enough so that both the if and the only if part can be satisfied.

Second solution. Consider the following graph: the trapezoid $ABCD$ with the bases $BC = 33$ and $AD = 34$ and the altitude $\sqrt{3}/2$ composed from 67 regular triangles with side 1, and the path with length 33 joining A and

D. It is clear that this graph can not be drawn on the plane satisfying the condition of the problem, but we can do it if an arbitrary vertex is removed.

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. Second day. 9 grade

Ratmino, 2016, August 1

5. (F.Nilov) The center of a circle ω_2 lies on a circle ω_1. Tangents XP and XQ to ω_2 from an arbitrary point X of ω_1 (P and Q are the touching points) meet ω_1 for the second time at points R and S. Prove that the line PQ bisects the segment RS.

   First solution. Let O be the center of ω_2. Since XO is the bisector of angle PXQ, we have $OR = OS$. Thus the right-angled triangles OPR and OQS are congruent by a cathetus and the hypothenuse, i.e. $PR = QS$ (fig.9.5). Since $\angle XPQ = \angle XQP$, we obtain that R and S lie at equal distances from the line PQ, which is equivalent to the required assertion.

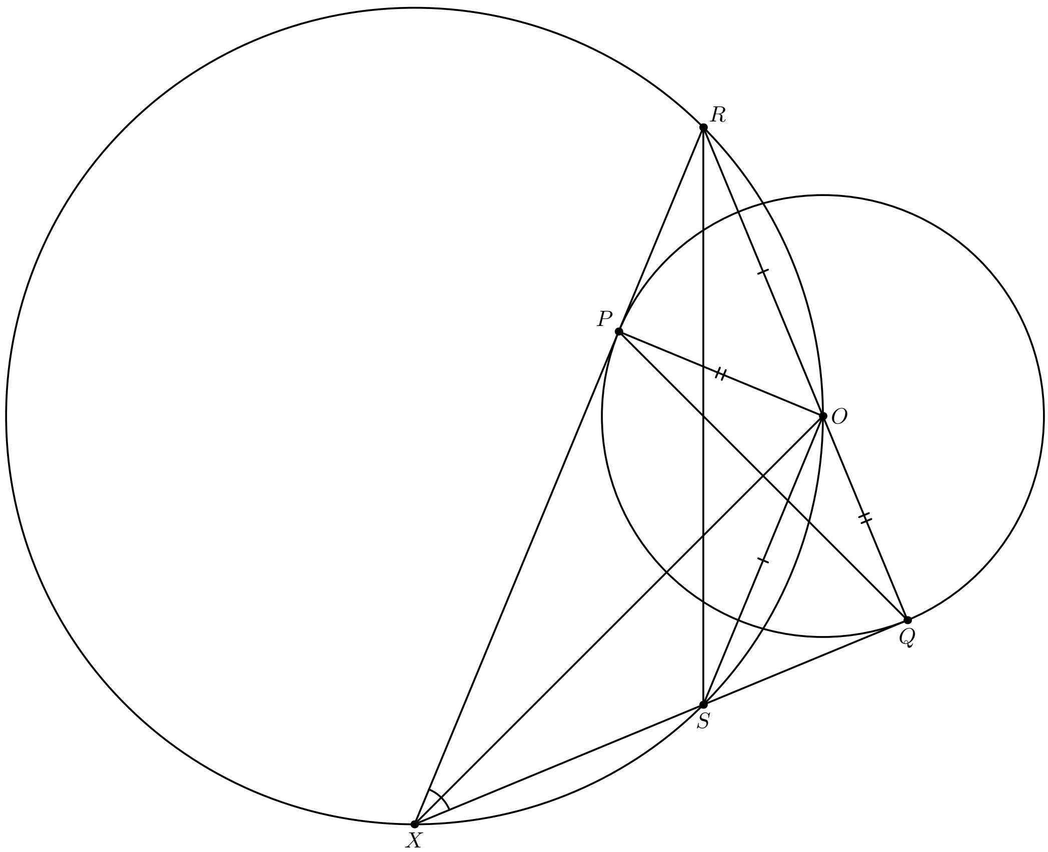

Fig. 9.5

   Second solution. Let O be the center of ω_2. Since XO bisects the angle PXQ, we obtain that O is the midpoint of arc RS. Hence the midpoint K of segment RS is the projection of O to RS. Therefore P, Q and K lie on the Simson line of point O.

6. (M.Timokhin) The sidelines AB and CD of a trapezoid $ABCD$ meet at point P, and the diagonals of this trapezoid meet at point Q. Point M on the smallest base BC is such that $AM = MD$. Prove that $\angle PMB = \angle QMB$.

First solution. Let the lines PM, QM meet AD at points X, Y respectively, and let U be the midpoint of AD. Since $AX : XD = BM : MC = YD : AY$, we obtain that $AX = YD$ and $XU = UY$ (fig. 9.6). Hence the perpendicular bisector UM of segment AD is also the bisector of isosceles triangle XMY, and BC is the bisector of angle PMQ.

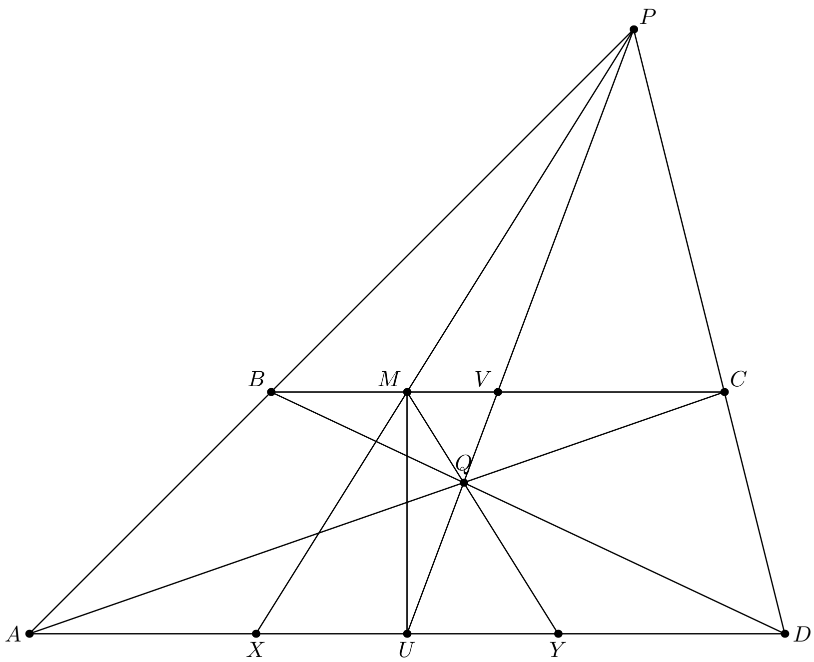

Fig. 9.6

Second solution. The line PQ passes through U and the midpoint V of segment BC (fig. 9.6) so that the quadruple P, Q, U, V is harmonic. Since the lines MU and MV are perpendicular they are the external and the internal bisectors of angle PMQ.

7. (A.Zaslavsky) From the altitudes of an acute-angled triangle, a triangle can be composed. Prove that a triangle can be composed from the bisectors of this triangle.

Solution. Let in a triangle ABC be $\angle A \geq \angle B \geq \angle C$. Then the altitudes h_a, h_b, h_c satisfy the inequality $h_a \leq h_b \leq h_c$, and the similar inequality holds for the bisectors l_a, l_b, l_c. Consider two cases.

1) $\angle B \geq 60^\circ$. Then $\angle A - \angle B \leq \angle B - \angle C$. Hence $h_c/l_c = \cos(\angle A - \angle B)/2 \geq h_a/l_a = \cos(\angle B - \angle C)/2$. Also $h_c/l_c > h_b/l_b$. Now from $h_c < h_a + h_b$ we obtain that $l_c < l_a + l_b$.

2) $\angle B \le 60^\circ$. Then since $\angle A < 90^\circ$, we have $\angle C > 30^\circ$. Thus $l_a \ge h_a = AC\sin\angle C > AC/2$ and $l_b > BC/2$. But l_c is not greater than the corresponding median, which is less than the half-sum of AC and BC. Therefore $l_c < l_a + l_b$.

Remark. Note that in the first case we did not use that the triangle is acute-angled, and in the second case we did not use that a triangle can be composed from the altitudes. But both conditions are necessary. An example of an obtuse-angled triangle, for which a triangle can be composed from the altitudes but not from the bisectors is constructed in the solution of problem 9.5 of VII Sharygin Olympiad.

8. (I.Frolov) The diagonals of a cyclic quadrilateral $ABCD$ meet at point M. A circle ω touches segments MA and MD at points P, Q respectively and touches the circumcircle of $ABCD$ at point X. Prove that X lies on the radical axis of circles ACQ and BDP.

First solution. The inversion with the center at X maps the lines AC and BD to the circles ω_1 and ω_2 intersecting at points X and M'. Furthemore this inversion maps ω to a line touching these circles at points P', Q' respectively. Finally it maps the circle $ABCD$ to a line parallel to $P'Q'$, meeting ω_1 at points A', C', and meeting ω_2 at points B', D' (fig. 9.8). Since M lies on the radical axis of circles ACQ and BDP, we have to prove that the radical axis of $A'C'Q'$ and $B'D'P'$ coincides with the line XM'.

Let K be the common point of XM' and $A'D'$. Since $A'K \cdot KC' = XK \cdot KM' = B'K \cdot KD'$, we obtain that K lies on the radical axis of circles $A'C'Q'$ and $B'D'P'$. Also the circle $A'C'Q'$ meets $P'Q'$ for the second time at the point symmetric to Q' about P', and the circle $B'D'P'$ meets it at the point symmetric to P' about Q'. Thus the degrees of the midpoint of $P'Q'$ lying on $M'X$, about these circles are also equal, and this completes the proof.

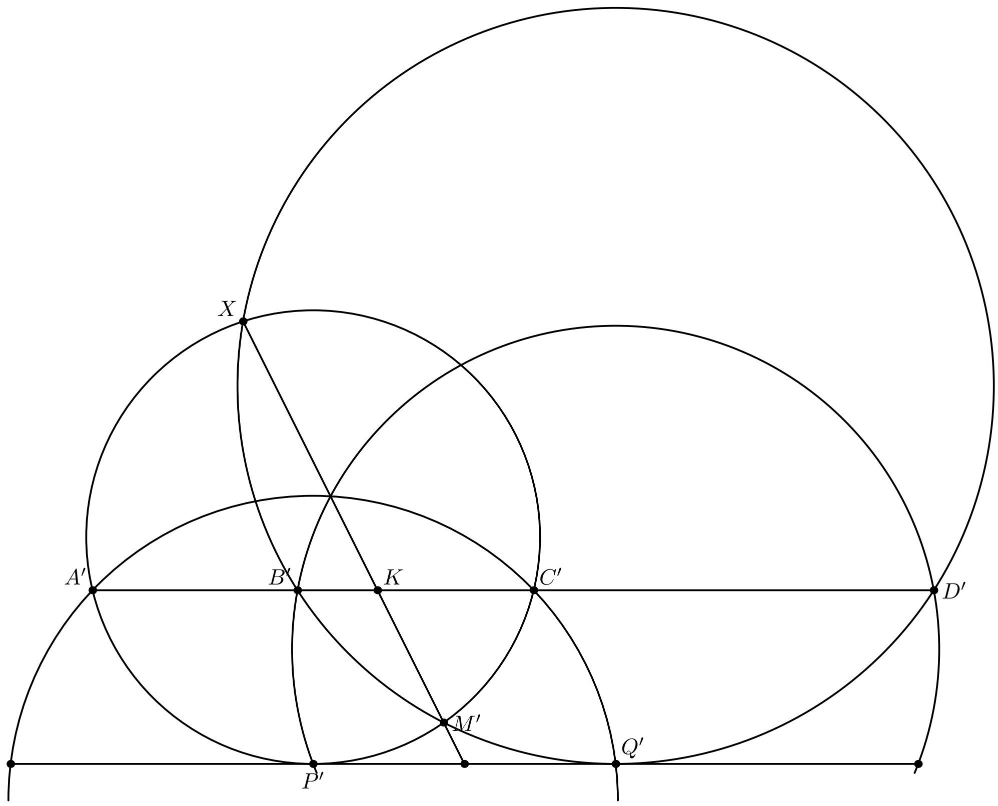

Fig. 9.8

Second solution. Let l be the tangent at X to the circle $ABCD$; and let l meet AC and BD at the points S and T respectively. Then SM is the radical axis of circles $ABCD$ and ACQ, ST is the radical axis of circles $ABCD$ and ω, i.e. S is the radical center of circles $ABCD$, ACQ and ω, hence SQ is the radical axis of circles ACQ and ω (because Q lies on the circles ACQ and ω). Similarly TP is the radical axis of circles BDP and ω. Therefore the common point G of SQ and TP is the radical center of circles ACQ, BDP and ω. On the other hand M is the radical center of circles ACQ, BDP and $ABCD$, i.e. MG is the radical center of circles ACQ and BDP, also MG passes through X, because G is the Gergonne point of triangle MST.

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 10 grade

Ratmino, 2016, July 31

1. V.Yasinsky A line parallel to the side BC of a triangle ABC meets the sides AB and AC at points P and Q, respectively. A point M is chosen inside the triangle APQ. The segments MB and MC meet the segment PQ at E and F, respectively. Let N be the second intersection point of the circumcircles of the triangles PMF and QME. Prove that the points A, M, and N are collinear.

First solution. Let P' and Q' be the second intersection points of the circle (PMF) with AB and of the circle (QME) with AC. We have $\angle MP'A = \angle MFP = \angle MCB$, so the point P' lies on the circle (BMC). Similarly, the point Q' also lies on the same circle. Therefore, we have $AP'/AQ' = AC/AB = AQ/AP$, which means that the powers of the point A with respect to the two given circles are equal. This yields that A lies on the line MN.

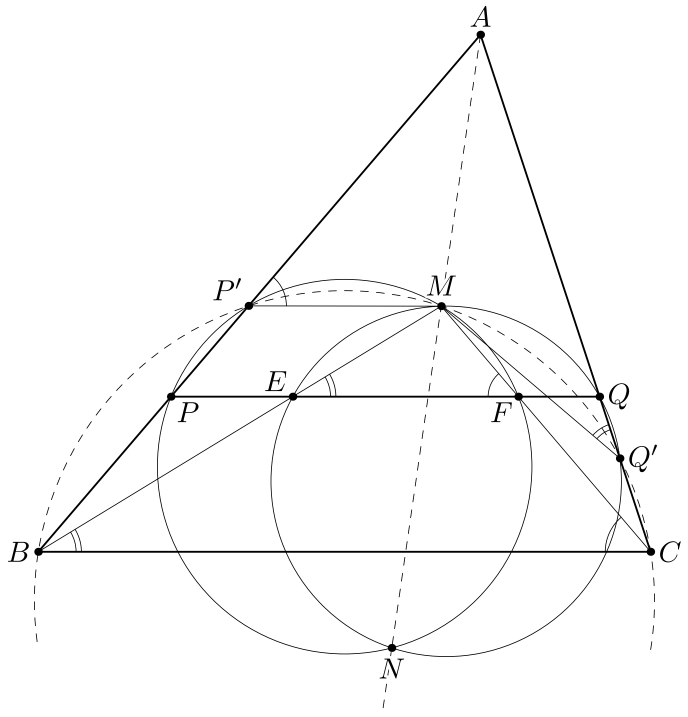

Fig. 10.1

Second solution. Let AM meet PQ and BC at points K and L respectively. Then $EK : FK = BL : CL = PK : QK$. Therefore, $PK \cdot FK = QK \cdot EK$ and both circles meet AM at the same point.

2. (P.Kozhevnikov) Let I and I_a be the incenter and the excenter of a triangle ABC; let A' be the point of its circumcircle opposite to A, and A_1 be the base of the altitude from A. Prove that $\angle IA'I_a = \angle IA_1I_a$.

   Solution. Since $\angle A_1AB = \angle CAA'$ and $\angle ACA' = 90°$, the triangles ACA' and AA_1B are similar. Hence $AA_a \cdot AA' = AB \cdot AC$. On the other hand $\angle AI_aC = \angle ABI$, thus the triangles AIB and ACI_a are similar and $AI \cdot AI_a = AB \cdot AC$.

   Let A_2 be the reflection of A_1 about the bisector of angle A. Then A_2 lies on AA' and as is proved above $AA_2 \cdot AA' = AI \cdot AI_a$. Therefore $IA_2A'I_a$ is a cyclic quadrilateral and $\angle IA'I_a = \angle IA_2I_a = \angle IA_1I_a$ (fig. 10.2).

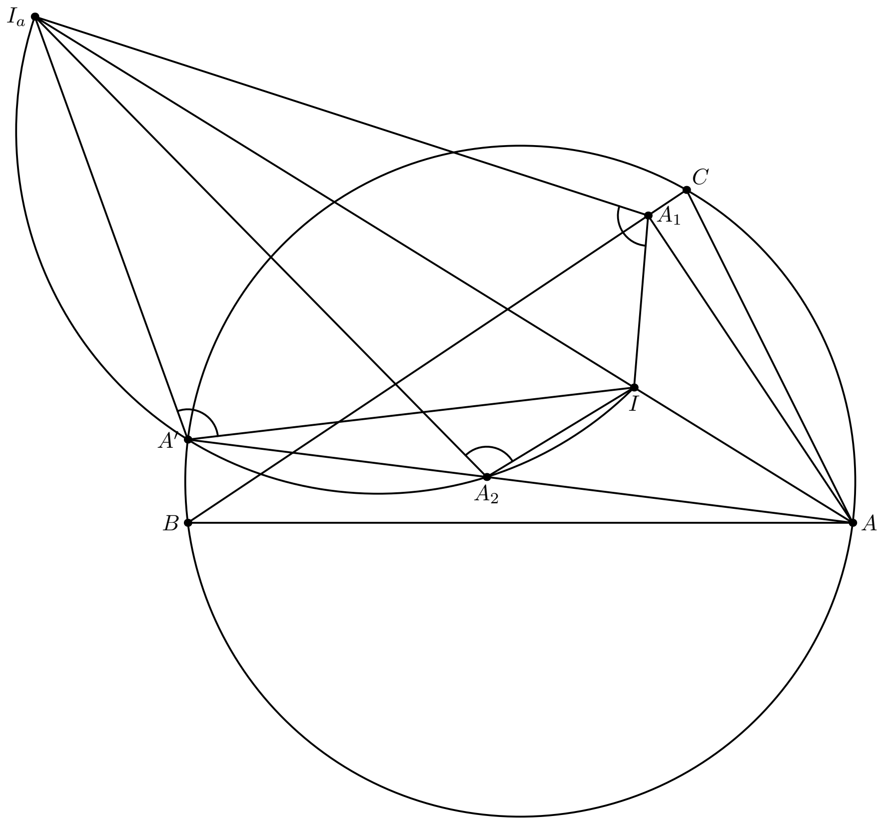

Fig. 10.2

3. (V.Kalashnikov) Let two triangles ABC and $A'B'C'$ have the common incircle and circumcircle, and let a point P lie inside both triangles. Prove that the sum of distances from P to the sidelines of ABC is equal to the sum of distances from P to the sidelines of $A'B'C'$.

   Solution. As is proved in the solution of problem 9.3, the locus of the points with constant sum of oriented distances to the sidelines of triangle ABC is a line perpendicular to OI, where O, I are the circumcenter and the incenter respectively. Also the sum of distances from I to the sidelines of both triangles is equal to $3r$, and the sum of the corresponding distances from O is equal to $R + r$ (the Carnot formula), where R and r are the radii of the circumcircle and the incircle. Therefore these sums are equal for all points of the plane.

Remark. The assertion remains true if we replace the triangles by two bicentric n-gons with an arbitrary n.

4. (N.Beluhov) Devil and Man are playing a game. Initially, Man pays some sum s and lists 97 triples (not necessarily distinct) $A_iA_jA_k$, $1 \le i < j < k \le 100$. After this Devil draws some convex 100-gon $A_1A_2\dots A_{100}$ of area 100 and pays the total area of 97 triangles $A_iA_jA_k$ to Man. For which maximal s this game is profitable for Man?

   Answer. For $s = 0$.

   First solution. *Lemma.* Let T be a set of at most $n-3$ triangles with the vertices chosen among those of the convex n-gon $P = A_1A_2\dots A_n$. Then the vertices of P can be coloured in three colours so that every colour occurs at least once, the vertices of every colour are successive, and T contains no triangle whose vertices have three different colours.

   Proof of the lemma. We proceed by induction on n.

   When $n = 3$, T is empty and the claim holds.

   Suppose $n > 3$. If A_1A_n is not a side of any triangle in T, then we colour A_1 and A_n in two different colours and all other vertices in the remaining colour.

   Now suppose that A_1A_n is a side of at least one triangle in T and the set U is obtained from T by removing all these triangles and replacing A_n by A_1 in all the remaining ones. By the induction hypothesis for the polygon $Q = A_1A_2\dots A_{n-1}$ and the set U, there is an appropriate colouring of the vertices of Q. By further colouring A_n in the colour of A_1, we get an appropriate colouring of P. $\square$

   Now imagine that Devil has chosen a convex 100-gon P of area 100 such that P is inscribed in a circle k, all vertices of P of colour i lie within the arc c_i of this circle with central angle ε°, and the midpoints of c_1, c_2, and c_3 form an equilateral triangle. When ε tends to zero, the areas of all triangles listed by the Man also tend to zero, and so does their sum.

Second solution. For each triple (i, j, k) let the vertex A_i be labelled by the number of sides covered by the angle $A_jA_iA_k$ (it is the same for all 100-gons), and do the same operation with A_j and A_k. The sum of these numbers is 100 for each triple, therefore the total sum is equal to $97 \cdot 100$, thus the sum in some vertex (for example A_1) is not greater than 97; from this we obtain that there exists a side A_kA_{k+1} not containing A_1 and such that the angles with vertex A_1 do not cover this side. Now the Devil can draw a 100-gon, in which the vertices $A_2, \ldots, A_{k-1}$ are close to A_k, and the vertices $A_{k+2}, \ldots, A_{100}$ are close to A_{k+1}, and make the areas of all 97 triangles arbitrary small.

XII Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. Second day. 10 grade

Ratmino, 2016, August 1

5. (A.Blinkov) Does there exist a convex polyhedron having equal numbers of edges and diagonals? (*A diagonal of a polyhedron is a segment between two vertices not lying in one face.*)

   Answer. Yes. For example each vertex of the upper base of a hexagonal prism is the endpoint of three diagonals joining it with the vertices of the lower base. Hence the common number of diagonals is 18 as well as the number of edges.

6. (I.I.Bogdanov) A triangle ABC is given. The point K is the base of an external bisector of angle A. The point M is the midpoint of arc AC of the circumcircle. The point N on the bisector of angle C is such that $AN \parallel BM$. Prove that M, N and K are collinear.

   First solution. Let I be the incenter. Then K, M, N lie on the sidelines of triangle BIC (fig. 10.6). We have $KB/KC = AB/AC$, $NC/IN = AC/AB' = (BC + AB)/AB$ (where B' is the base of bisector from B), $MI/MB = MC/MB = AB'/AB = AC/(AB + BC)$ (the second equality follows from the similarity of triangles BMC and BAB'). By the Menelaus theorem we obtain the required assertion.

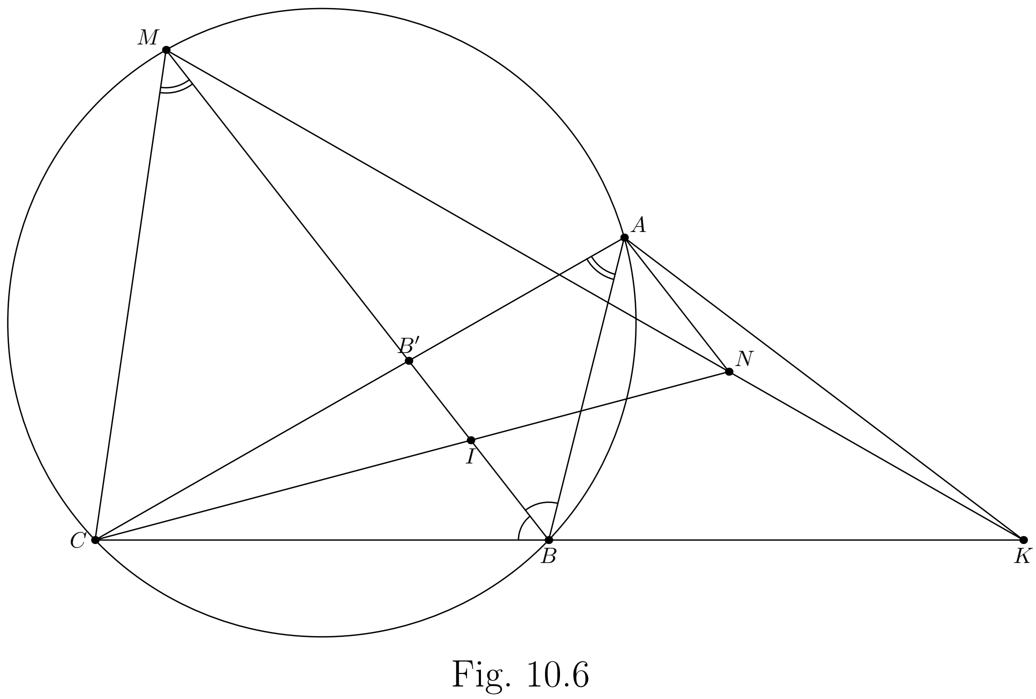

Fig. 10.6

Second solution. Note that $\angle MAC = \angle MBC = \angle ABM = \angle BAN$, i.e. the lines AI and AK are the internal and the external bisector of triangle AMN. Let AI meet MN and BC at points P and Q respectively, and let AK meet MN at K'. Then the quadruples (B, C, K, Q) and (M, N, K', P) are harmonic, therefore projecting MN to BC from I, we obtain that K' coincides with K.

7. (A.Zaslavsky) Restore a triangle by one of its vertices, the circumcenter and the Lemoine point. (*The Lemoine point is the common point of the lines symmetric to the medians about the correspondent bisectors.*)

   First solution. Since the vertex A and the circumcenter O are given, we can construct the circumcircle. Let XY be the chord of this circle with the midpoint at the Lemoine point L, let UV be the diameter parallel to this chord, and let the diagonals of the trapezoid with bases XY and UV meet at point K. Consider a transformation that maps each point P of the circumcircle to the second common point P' of the circle and the line KP. This transformation preserves the cross-ratios, thus it can be extended to a projective transformation of the plane. Since this transformation maps L to O, it maps the triangle in question to a triangle with coinciding Lemoine point and circumcenter. This triangle is regular. From this we obtain the following construction.

   Draw line AK and find its second common point A' with the circumcircle. Inscribe a regular triangle $A'B'C'$ into the circle and find the second common points B, C of lines BK, CK with the circle. Then ABC is the required triangle.

   Second solution. We use the following assertion.

   Lemma. Let a triangle ABC and a point P be given. An inversion with center A maps B, C, P to B', C', P' respectively. Let the circle $B'C'P'$ meet AP for the second time at Q. Then the similarity transforming the triangle $AC'B'$ to ABC maps Q to the point isogonally conjugated to P.

   The assertion of this lemma clearly follows from the equalities $\angle ABP = \angle B'P'A = \angle B'C'Q$.

   Let us return to the problem. Let an inversion with center A map L and the circumcircle to L' and line l respectively. Let AL meet l at point T, and let point M divide the segment AT in ratio $2 : 1$. Then M is the centroid of triangle $AB'C'$, where B', C' are the images of B and C. By the lemma M

lies on the circle $B'C'L'$, therefore $KB'^2 = KC'^2 = KM \cdot KL'$. Thus we can construct B', C', and B, C.

8. (S.Novikov) Let ABC be a nonisosceles triangle, let AA_1 be its bisector, and let A_2 be the touching point of BC with the incircle. The points B_1, B_2, C_1, C_2 are defined similarly. Let O and I be the circumcenter and the incenter of the triangle. Prove that the radical center of the circumcircles of triangles AA_1A_2, BB_1B_2, CC_1C_2 lies on OI.

   First solution. Let A' be the midpoint of an arc BC not containing A. Since the inversion with center A' and radius $A'B$ transposes the line BC and the circumcircle, it maps A_1 and A_2 to A and the common point A'' of $A'A_2$ and the circumcircle. Therefore the points A, A_1, A_2 and A'' are concyclic. Furthermore since $OA' \parallel IA_2$, the lines OI and $A'A_2$ meet at the point K which is the homothety center of the circumcircle and the incircle (fig. 10.8). Hence the degree of K wrt the circle AA_1A_2 is equal to

$$(\vec{KA_2}, \vec{KA''}) = \frac{r}{R}(\vec{KA'}, \vec{KA''}) = -\frac{r^3R}{(R-r)^2},$$

   because $(\vec{KA'}, \vec{KA''})$ is the degree of K wrt the circumcircle, equal to $-R^2r^2/(R-r)^2$.

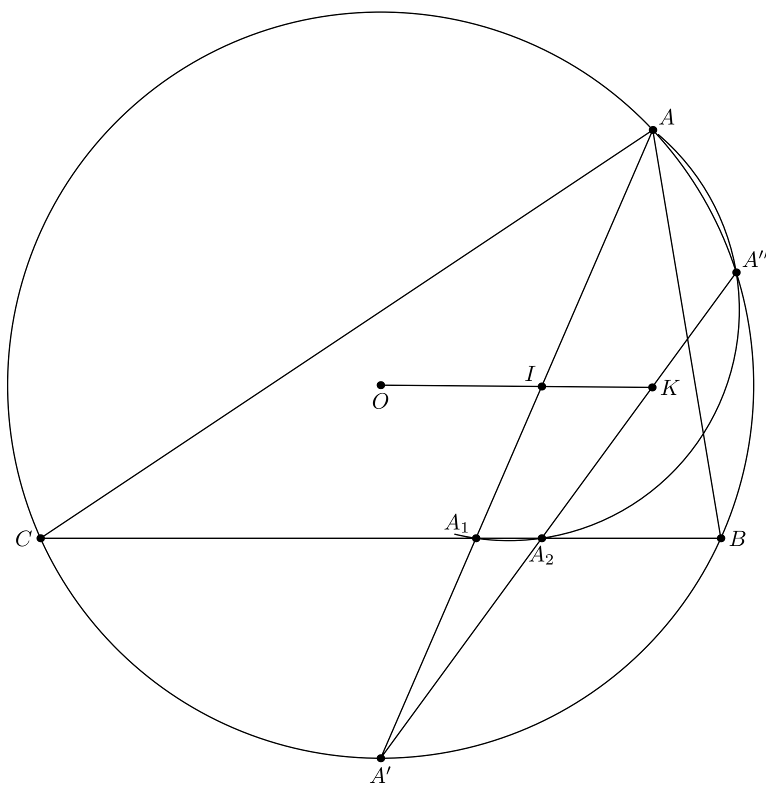

Fig. 10.8

Clearly the degrees of K wrt the circles BB_1B_2 and CC_1C_2 are the same, i.e. K is the radical center.

Second solution. Let A', B', C' be the midpoints of the arcs BC, CA, AB. Then the triangles $A'B'C'$ and $A_2B_2C_2$ are homothetic with a positive coefficient and center K, i.e. $KA_2/A'A_2 = KB_2/B'B_2 = KC_2/C'C_2 = k$. For the points of line $A'A_2$ consider the difference of the degrees wrt AA_1A_2 and the incircle. This is a linear function. In A_2 this function is equal to zero, and in A' it is equal to r^2 because $A'A_1 \cdot A'A = A'B^2 = A'I^2$. Thus in K this difference is equal to $-kr^2$. Two similar differences in K are also equal to $-kr^2$, and we obtain the required assertion.